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ON THE STABILITY OF FLOW OF WEAKLY COMPRESSIBLE GAS IN A PIPE OF MODEL ROUGHNESS

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The problem of stability of weakly compressible gas flow in a rough pipe is considered in linear approximation with respect to stationary waves. It is assumed that the laminar velocity profile, roughness, and velocity perturbations are axisymmetric. The critical Reynolds number is obtained in the form of function of the root-mean-square roughness amplitude.

In the considered flow an undamped stationary wave is formed at lowest Reynolds numbers as the result of interaction between a convective unsteady vortex wave and the acoustic wave which propagates upstream [1]. This is due to that the remaining waves propagating upstream are damped considerably quicker than the acoustic wave. Because of this we consider here the interaction between the vortex and the acoustic waves. In order to eliminate the effects produced by sound reflection from the pipe inlet boundary conditions that ensure the absence or reasonable reduction of sound reflection there. According to current concepts the Poiseuille flow is convectively stable with respect to axisymmetric perturbations. Hence it is possible to stipulate the damping of perturbations in the downstream flow.

1. The equations that will be considered below must define the vortex and the acoustic waves and, also, the interaction between these at $M \ll 1$ (M is the Mach number). Assuming that pipe wall has adiabatic properties, the Navier-Stokes equations in dimensionless form are

$$\frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad}(\mathbf{v}_0 \cdot \mathbf{v}) + \operatorname{rot} \mathbf{v}_0 \times \mathbf{v} + \operatorname{rot} \mathbf{v} \times \mathbf{v}_0 = -\operatorname{grad} P + \frac{1}{R} \Delta \mathbf{v} \tag{1.1}$$
$$\operatorname{div} \mathbf{v} = -M^2 \partial P / \partial t$$

where the basic dimensions are the flow rate and the pipe mean radius; $\mathbf{v}_0$ is the steady solution of the Navier-Stokes equation. Terms of order M^2 / R are omitted in Eqs. (1.1), and in the equation of continuity the term $M^2 (\mathbf{v}_0 \cdot \operatorname{grad} P)$ is omitted. This is justified since for an acoustic wave $P \sim pMP$, while $\partial P / \partial t \sim pP$ ($P = Pe^{pt}$).

We seek a solution of system (1.1) in the form of a standing wave

$$P = Pe^{pt}, \quad \mathbf{v} = \mathbf{v}e^{pt} \tag{1.2}$$

$$p\mathbf{v} + \operatorname{grad}(\mathbf{v}_0 \cdot \mathbf{v}) + \operatorname{rot} \mathbf{v}_0 \times \mathbf{v} + \operatorname{rot} \mathbf{v} \times \mathbf{v}_0 = -\operatorname{grad} P + \frac{1}{R} \Delta \mathbf{v} \tag{1.3}$$
$$\operatorname{div} \mathbf{v} = -M^2 pP; \quad \mathbf{v}(X, r(X)) = 0; \quad X \to \infty, \ \mathbf{v} \to 0$$

where X and r are the longitudinal and radial coordinates, respectively, and $r(X)$ is the coordinate of the pipe wall that represents a stationary random function with characteristic amplitude a, scale Δ, and spectral density

$$s(\omega) = \frac{a^2\Delta}{2}\left(\eta\left(|\omega| - \frac{1}{\Delta}\right) - \eta\left(|\omega| - \frac{2}{\Delta}\right)\right), \quad \eta(\omega) = \begin{cases} 1, & \omega > 0 \\ 0, & \omega < 0 \end{cases} \tag{1.4}$$

The homogeneous boundary conditions at the pipe inlet cross section that define the absence of sound reflection there are given in Sect. 2.

The state of stability is understood here in the meaning given in [2], i.e. if for all p_i, defined by the problem (1.3) the condition $\operatorname{Re} p_i < 0$ is satisfied, the laminar flow is called stable.

The problem is considered with the following constraints on a and Δ:

$$|\operatorname{rot} \mathbf{v}_0|\, \Delta^2 / \nu \ll 1, \quad |r'(X)| \leqslant a / \Delta \ll 1 \tag{1.5}$$

The first condition implies that the flow over the wall irregularities is viscous (ν is the kinematic viscosity coefficient, rot $\mathbf{v}_0$, and Δ are here dimensional). The second condition is unrelated to physical constraints but makes possible a considerable simplification of the problem.

2. It is assumed that Eq. (1.3) can be expressed in terms of coordinates

$$\theta, \quad x = \varphi(X, r), \quad y = \psi(X, r)$$
$$(\partial\psi / \partial r = -rv_X, \quad \psi(X, 0) = 1/2, \quad \psi(X, r(X)) = 0)$$

where θ is the angle, and φ and ψ are the potential and the stream function of the corresponding stationary problem for an incompressible inviscid medium.

We compose vector $\mathbf{A} = \{A^1, A^2, A^3, A^4\}$ as follows (h_θ, h_x, and h_y are Lamé coefficients):

$$h_\theta h_y v_x = A^1, \quad P = A^2, \quad h_\theta h_x v_y = A^3, \quad (1 - 2y)\frac{\partial A^1}{\partial y} - \frac{\partial A^3}{\partial x} = A^4$$

In the new notation problem (1.3) may be written as

$$\partial\mathbf{A} / \partial x = L\mathbf{A} = H\mathbf{A} + H_1\mathbf{A}, \quad H = \lim L, \quad a \to 0 \tag{2.1}$$
$$y = 0, \quad A^1 = A^3 = 0; \quad x \to \infty, \quad \mathbf{A} \to 0$$
$$x = 0, \quad \partial A^i/\partial x - \lambda_3 A^i = 0, \quad i = 1, 2$$

The form of operators H and H_1 is determined by formulas (1.3) and the form of $\mathbf{A}$; λ_3 is the wave number of the acoustic wave that propagates upstream.

The solution of problem (2.1) is sought in the form of superposition of eigenvectors of operator H

$$\mathbf{A} = \sum c_i \mathbf{A}_i, \quad H\mathbf{A}_i = \lambda_i \mathbf{A}_i; \quad y = 0, \quad A_i^1 = A_i^3 = 0 \tag{2.2}$$

From (2.1) we have

$$c_i' - \lambda_i c_i = c_m (H_1\mathbf{A}_m, \mathbf{B}_i) / (\mathbf{A}_i, \mathbf{B}_i) \tag{2.3}$$
$$x = 0, \ c_k = 0; \quad x \to \infty, \ c_l \to 0$$

where the subscript k denotes waves propagating downstream and B_i are eigenvectors of the operator conjugate of H.

As in [3], we consider the sought solution of the problem as the limit solution of problem (2.3) for

$$k = 1, 2, \ldots, N, \quad l = -N, -N+1, \ldots, N, \quad N \to \infty \qquad (2.4)$$

A problem of the form (2.3), (2.4) was considered in [4] whose results will be used subsequently. To write the equation for the eigenvalue curve of problem (2.3), (2.4) it is necessary to know two wave numbers λ_1 and λ_3 which in the considered interval satisfy the condition

$$\mathrm{Re}\,(\lambda_1 - \lambda_3) = 0 \qquad (2.5)$$

and the expressions for the coupling coefficients

$$(H_1\mathbf{A}_3, \mathbf{B}_1) / (\mathbf{A}_1, \mathbf{B}_1), \quad (H_1\mathbf{A}_1, \mathbf{B}_3) / (\mathbf{A}_3, \mathbf{B}_3) \qquad (2.6)$$

In the considered case λ_1 is the wave number of the unstable vortex wave.

The results obtained in the considered problem using the formalism of [4] are only valid when the vortex and acoustic waves differ only slightly in the interaction region (at distances of order Δ from the wall) for $H_1 = 0$ and $H_1 \neq 0$, which means that the wave must be of the form

$$c_i \exp\left(\int \lambda_i\, dx\right)(\mathbf{A}_i + \mathbf{D}_i), \quad D_i^k = \varepsilon f_i^k A_i^k, \quad i = 1, 3 \qquad (2.7)$$

where f_i^k are stationary random functions of amplitude or order unity and correlation dimension of order Δ. When $k = 3$ condition (2.7) need not be satisfied, since the third column of operator H_1 consists of zeros.

The constraint on ε can be established on the following considerations. We represent the expression in the right-hand side of system (2.3) in the form

$$(H_1 \Sigma c_m \mathbf{A}_m, \mathbf{B}_k) / (\mathbf{A}_k, \mathbf{B}_k), \quad k = 1, 3 \qquad (2.8)$$

If the sum in (2.8) corresponds to the considered here wave, we have

$$c_i \exp\left(\int \lambda_i\, dx\right)[(\mathbf{A}_i, \Pi_1\mathbf{B}_k) + (\mathbf{D}_i, \Pi_1\mathbf{B}_k)] / (\mathbf{A}_k, \mathbf{B}_k); \quad i = 1, 3;\ k = 1, 3$$

where Π_1 is the conjugate operator of H_1.

Prior to proceeding any further we shall point out that the amplitude of low-frequency harmonics $\Pi_1\mathbf{B}_k$ is proportional to $a\sqrt{\Delta a} / \Delta$, and that of high-frequency harmonics is proportional to $a\sqrt{\Delta}$ (see Sectn. 4). The amplitude of low-frequency harmonics in $(\mathbf{A}_i, \Pi_1\mathbf{B}_k)$ is proportional to $a\sqrt{\Delta a} / \Delta$, and in $(\mathbf{D}_i, \Pi_1\mathbf{B}_k)$ is proportional to $\varepsilon a\sqrt{\Delta}$. This yields the constraint

$$\varepsilon \ll a / \Delta \qquad (2.9)$$

Since the mean pipe radius is considerably greater than the dimensions of wall surface imperfections, it is possible to consider the problem near the wall as a plane one in which $h_\theta = 1,\ h_x = h_y = h^{-1}$. The equation for

$$Q = \int A^1\, dy$$

with condition (1.5) is of the form

$$Th^2TQ = 0, \quad T = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \quad y = 0,\ Q = \frac{\partial Q}{\partial y} = 0; \quad y \to \infty,\ \frac{2Q}{y^2} \to 1$$

where the coordinates are stretched $1 / \Delta$ -times and parameter A^4 is normalized on unity when $y \to \infty$, and y is measured from the wall.

Since $Th^2 \sim (a^2 / \Delta^2)$, hence $Q = {}^1/_2 y^2 (1 + o(a^2 / \Delta^2))$, $A^1 = y(1 + o(a^2 / \Delta^2))$, and $A^4 = 1 + o(a^2 / \Delta^2)$, which shows that condition (2.9) is satisfied.

3. Let us derive the expressions for the eigenvectors appearing in (2.6). In accordance with formulas (2.1) and (2.2) the eigenvectors are calculated on the assumption of smooth pipe wall. The derivatives of u with respect to x (u is the longitudinal velocity component in the stationary solution of the Navier-Stokes equation for a pipe with smooth wall) are omitted in calculations. Moreover, compressibility is neglected in the calculation of $\mathbf{A}_1$, and $\mathbf{B}_1$ while in the calculation of $\mathbf{A}_3$, and $\mathbf{B}_3$ we neglect convection terms, since the respective corrections are small, namely

$$A_1^1 \approx \frac{\partial \varphi_1}{\partial y}, \quad A_1^2 \approx \left(-\frac{p}{\lambda_1}\frac{\partial}{\partial y} - u\frac{\partial}{\partial y} + u' + \frac{1}{\lambda_1 R}\frac{\partial}{\partial y}\left((1-2y)\frac{\partial^2}{\partial^2 y} + \lambda_1^2\right)\right)\varphi_1$$

$$A_1^3 \approx -\lambda_1\varphi_1, \quad A_1^4 \approx (1-2y)\frac{\partial^2\varphi_1}{\partial y^2} + \lambda_1^2\varphi_1$$

$$A_3^1 \approx u_3 \sim M, \quad A_3^2 \approx 1, \quad A_3^3 \sim M^2\left|\frac{p}{R}\right|^{1/2}, \quad A_3^4 \approx (1-2y)\frac{\partial u_3}{\partial y}$$

where φ_1 is the eigenfunction of the vortex wave whose characteristic scale is of order $(u')^{-3/4}R^{-1/4}$, and the equation for φ_1 appears in [5]. The expression for u_3 whose characteristic measure is of order $|pR|^{-1/2} \sim (u')^{-3/4}R^{-1/4}$; is given in [1].

$\mathbf{B}_1$, and $\mathbf{B}_3$ are determined by formulas

$$\bar\lambda_i \mathbf{B}_i = \Pi \mathbf{B}_i; \quad y = 0, \quad B_i^2 = B_i^4 = 0$$

$$B_1^1 \approx \frac{1}{\bar\lambda_1}\frac{\partial}{\partial y}\left((1-2y)\left(\frac{\partial^2(1-2y)}{\partial y^2} + \bar\lambda_1^2 - \bar p R\right)\right)\varphi_2, \quad B_1^2 \approx R\frac{\partial(1-2y)\varphi_2}{\partial y}$$

$$B_1^3 \approx \left(\bar\lambda_1 Ru - \bar\lambda_1^2 - \frac{\partial^2(1-2y)}{\partial y^2}\right)\varphi_2, \quad B_1^4 \approx \bar\lambda_1\varphi_2$$

$$B_3' \approx 1, \quad B_3^2 \approx u_3, \quad B_3^3 \sim M \left| \frac{p}{R} \right|^{1/2}, \quad B_3^4 \sim \frac{M^2}{R} \left| \frac{p}{R} \right|^{1/2}$$

where Π is the conjugate operator of H and φ_2 satisfies the equation conjugate to the equation for φ_1.

4. The definitionof roughness (1.4), conditions (1.5), and the eigenvector properties indicated in Sectn. 3 imply that in calculating H_1 it is possible to disregard convection terms. Representing Eq. (1.3) in curvilinear coordinates (see Sectn. 2) and setting $h_\theta = 1,\ h_x = h_y = h^{-1}\ (\Delta \ll 1)$ we obtain

$$H_1 = \left\| \begin{matrix} 0 & M^2 p (1 - h^{-2}) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R} \frac{\partial}{\partial y} (h^2 - 1) \\ 0 & 0 & 0 & 0 \\ 0 & R (1 - h^{-2}) \frac{\partial}{\partial y} & 0 & (h^{-2} - 1) \frac{\partial}{\partial x} (h^2 - 1) \end{matrix} \right\| \tag{4.1}$$

Using formulas of Sectn. 2, formula (4.1), and omitting small terms we obtain expressions (2.6) of the form

$$(H_1 \mathbf{A}_3, \mathbf{B}_1) / (\mathbf{A}_1, \mathbf{B}_1) \approx M W_1 F, \quad W_1 = \sqrt{-ipR}\, \frac{\partial^2 \bar{\varphi}_2}{\partial y^2} \bigg|_{y=0} \Big/ (\mathbf{A}_1 \mathbf{B}_1) \tag{4.2}$$

$$(H_1 \mathbf{A}_1, \mathbf{B}_3) / (\mathbf{A}_3, \mathbf{B}_3) \approx W_2 F, \quad W_2 = \sqrt{\frac{ip}{R}}\, \frac{\partial^2 \varphi_1}{\partial y^2} \bigg|_{y=0}, \quad F = \int_0^\infty (h^2 - 1)\, dy$$

where $F(x)$ is a stationary random process whose intensity I we express in terms of roughness characteristics. Since $\Delta \ll 1$, we determine F by considering the plane problem which is the same as that of determination of the potential over the conducting rough surface. We seek a solution in the form of series in the small ratio a / Δ c with an accuracy within the first term ($f(\alpha)$ is the Fourier transform $r(X)$)

$$1 - r \approx y + y_1, \quad y_1 = \int f(\alpha) \exp(i\alpha x - |\alpha| y)\, d\alpha \tag{4.3}$$

$$X \approx x + x_1, \quad x_1 = \int f(\alpha) \operatorname{sign}(\alpha) \exp(i\alpha x - |\alpha| y)\, d\alpha$$

$$h^2 - 1 = \left(\left(\frac{\partial r}{\partial y} \right)^2 + \left(\frac{\partial X}{\partial y} \right)^2 \right)^{-1} \approx -2 \frac{\partial y_1}{\partial y} + 3 \left(\frac{\partial y_1}{\partial y} \right)^2 - \left(\frac{\partial x_1}{\partial y} \right)^2$$

Using formulas (1.4) and (4.3) we obtain

$$I \approx 25 \int_0^\infty s^2(\omega)\, \omega^2\, d\omega \approx 15 a^3 \frac{a}{\Delta} \tag{4.4}$$

5. Using formula (4.2) for the coupling constant (2.5) we obtain, in conformity with [4], the equation of the curve of eigenvalues of problem (2.3), (2.4). As stated in Sect. 2, this problem is in some sense equivalent to problem (1.3).

The equation for the eigenvalue curve is of the form

$$M^2I^2d^2 = 1, \quad d = \exp\left(\int_{x_1}^{x_2} \lambda_2\,dx\right)|W_1(x_1)||W_2(x_2)|\left(\frac{\partial\lambda_r}{\partial x}\bigg|_{x_1}\frac{\partial\lambda_r}{\partial x}\bigg|_{x_2}\right)^{-1/4} \qquad (5.1)$$

$$\lambda_r(x_1) = \lambda_r(x_2) = 0, \quad x_2 > x_1, \quad \lambda_r = \operatorname{Re}\lambda_1$$

Equation (5.1) was solved numerically on a computer. First, d was calculated with fixed R and p and, then, $b = \max_p d^2$ ($\operatorname{Re} p = 0$) was determined for fixed R. This made it possible to determine the critical Reynolds number R_* by specifying M and I. Parameters λ_1 and φ_1 were determined by solving the linear spectral problem of the form defined in [5]. The computation program in [6] was taken as the basis for numerical solution of this problem.

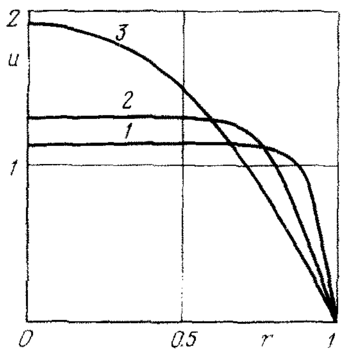

Fig. 1

The velocity profile at the pipe inlet was calculated using equations of the boundary layer type of the form given in [5], and the numerical scheme developed in [7]. The Blasius solution was taken as the definition of the initial profile ($z = X / R = 0$). The obtained velocity profiles are shown in Fig. 1 for $z = 16 \cdot 10^{-4}$, $4 \cdot 10^{-3}$, and $15 \cdot 10^{-2}$ by curves 1, 2, and 3, respectively.

Parameter b was determined in the interval $10 > 10^{-4}R > 3$ in the form of the analytic approximation $^1/_2 \lg b = 2 + 10^{-4}R$ whose error in that interval does not exceed 3%.

From this, using (4.4) we obtain

$$R_* = [-3.2 - \lg M - 3\lg a - \lg(a / \Delta)] \cdot 10^4$$

This result is valid in the region where conditions (1.5) are satisfied. As a rough approximation it can be presented in the form $R_*\Delta^2 < 1$.

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